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TECHNICAL NOTE

Non-Fourier heat conduction in a finite medium under periodic surface thermal disturbance—II. Another form of solution

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INTRODUCTION

Since Cattaneo and Vernotte independently postulated the hyperbolic heat conduction equation in 1958, much effort has been made to obtain solutions to this equation for different conditions, and to develop mathematical and numerical techniques that would accurately predict the non-Fourier temperature profiles for a wide range of physical geometric and boundary conditions. Most of the previous works [1–3] were performed for a pulse heat flux or a sudden temperature change, while the work for a periodic flux in a finite medium is seldom found in literature. Recently, the authors presented an analytical solution of the hyperbolic heat conduction equation in a finite medium under periodic surface heating [4]. This note considers the same situation with the preceding one, but here because a different mathematical treatment is used in the derivation, a quite different form of temperature solution is obtained. The numerical calculation shows that the present solution actually predicts the same temperature profiles as those in the preceding paper [4].

TEMPERATURE SOLUTION

As with ref. [4], we consider a one-dimensional heat conduction in an insulated finite medium with a thickness of L, constant thermal properties, and initial temperature distribution $T(x,0) = 0$. From time $t = 0$ the external surface at $x = 0$ is exposed to a periodic heat flux with amplitude q_0 and frequency ω. In this situation, the governing equation and boundary and initial conditions are

$$a\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2} \tag{1}$$

$$-\lambda\frac{\partial}{\partial x}T(0,t) = \tau_0\frac{\partial}{\partial t}q(0,t) + q(0,t)$$

$$q(0,t) = q_0\cos\omega t \tag{2a}$$

$$\frac{\partial}{\partial x}T(L,t) = 0 \tag{2b}$$

$$T(x,0) = 0,\quad \frac{\partial}{\partial t}T(x,0) = 0\quad q(x,0) = 0. \tag{3}$$

Applying the Laplace transformation to equation (1), by taking into account the initial condition (3), yields the following subsidiary equation:

$$a\bar{T}'' - (s - \tau_0 s^2)\bar{T} = 0 \tag{4}$$

with conditions

$$\frac{\partial}{\partial x}\bar{T}(0,s) = -\frac{q_0}{\lambda}(1+\tau_0 s)\bar{q}(s) \quad\text{and}\quad \frac{\partial}{\partial x}\bar{T}(L,s) = 0 \tag{5}$$

where $\bar{T}(x,s) = L[T(x,t)]$ and $\bar{q}(s) = L[\cos\omega t]$ are the Laplace transform of $T(x,t)$ and $\cos\omega t$, respectively.

The solution of equation (4) with respect to the condition (5) is

$$\bar{T}(x,s) = \frac{q_0}{\lambda}F_1(x,s)F_2(s) \tag{6}$$

where

$$F_1(x,s) = (1+\tau_0 s)\frac{e^{-rx} + e^{-r(2L-x)}}{r(1-e^{-2rL})}\qquad F_2(s) = \bar{q}(s) \tag{7}$$

and

$$r = \sqrt{(s+\tau_0 s^2)/a}.$$

For convenience in subsequent derivation, the following functions are introduced:

$$f_1(x,t) = L^{-1}[F_1(x,s)],$$
$$f_2(t) = L^{-1}[F_2(s)] = \cos\omega t. \tag{8}$$

Then by performing inverse transform on equation (6) and according to the property of inverse transform, the temperature response is obtained as

$$T(x,t) = L^{-1}[\bar{T}(x,s)] = \frac{q_0}{\lambda}L^{-1}[F_1(x,s)F_2(s)]$$
$$= \frac{q_0}{\lambda}f_1^{*}f_2(x,t) \tag{9}$$

where

$$f_1^{*}f_2(x,t) = \int_0^t f_1(x,t')f_2(t-t')\,dt'. \tag{10}$$

The function $f_1(x,t)$ can be derived by introducing the

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NOMENCLATURE

a	thermal diffusivity
Fo	Fourier number, at/L^2
Fo_1	Fourier number based on frequency of heating flux, $a/\omega L^2$
$I_0(\ )$	modified Bessel function of first kind of zeroth order
L	thickness of medium
q	heat flux
q_0	amplitude of periodic heating flux
s	Laplace variable
T	temperature
$\bar{T}$	Laplace transformation of T
t	time
$u(\ )$	Heaviside unit step function
V	dimensionless temperature defined by equation (17)
Ve	Vernotte number, $\sqrt{a\tau_0}/L$
X	dimensionless spatial variable, x/L
x	spatial variable.

Greek symbols

λ	thermal conductivity
τ_0	relaxation time
ω	frequency of periodic heating flux.

$$\begin{aligned} H(x,t) &= L^{-1}[\bar{H}(x,s)] \\ &= \sqrt{\frac{a}{\tau_0}}\,e^{-t/2\tau_0}\sum_{n=0}^{\infty}\left[I_0\left(\frac{1}{2\tau_0}\sqrt{t^2-(2nL+x)^2\frac{\tau_0}{a}}\right)u\left(t-\sqrt{\frac{\tau_0}{a}}(2nL+x)\right)\right. \\ &\quad \left. + I_0\left(\frac{1}{2\tau_0}\sqrt{t^2-(2nL+2L-x)^2\frac{\tau_0}{a}}\right)u\left(t-\sqrt{\frac{\tau_0}{a}}(2nL+2L-x)\right)\right] \end{aligned} \tag{13}$$

following series expansion to $F_1(x,s)$ in equation (7) (this mathematical treatment is different from that in ref. [4]):

$$(1-\exp(-2rL))^{-1} = \sum_{n=0}^{\infty}\exp(-2rnL)$$

then $F_1(x,s)$ can be written in form of

$$F_1(x,s) = \bar{H}(x,s)+\tau_0 s\bar{H}(x,s) \tag{11}$$

where

$$\bar{H}(x,s) = \sum_{n=0}^{\infty}\frac{1}{r}[e^{-r(2nL+x)}+e^{-r(2nL+2L-x)}]. \tag{12}$$

The source function of $\bar{H}(x,s)$ can be found in a table of inverse Laplace transformation, that is

where $I_0(\ )$ is the modified Bessel function of the first kind of order zero and $u(\ )$ is the Heaviside unit step function. Performing inverse Laplace transformation on equation (11) yields

$$\begin{aligned} f_1(x,t) &= L^{-1}[\bar{H}(x,s)]+\tau_0 L^{-1}[s\bar{H}(x,s)] \\ &= H(x,t)+\tau_0\frac{\partial H(x,t)}{\partial t}. \end{aligned} \tag{14}$$

After substituting equations (14) and (10), equation (9) takes the form

$$T(x,t) = \frac{q_0}{\lambda}\int_0^t\left[H(x,t')+\tau_0\frac{\partial H(x,t')}{\partial t'}\right]\times\cos\omega(t-t')\,dt'. \tag{15}$$

From equation (13) and by a series of manipulations, the non-Fourier temperature distribution inside the finite medium is obtained as

$$\begin{aligned} \frac{T(x,t)}{q_0/\lambda} &= \sqrt{a\tau_0}\,e^{-t/2\tau_0}\sum_{n=0}^{\infty}\left\{\begin{aligned}&I_0\left[\frac{1}{2\tau_0}\sqrt{t^2-(2nL+x)^2\frac{\tau_0}{a}}\right]u\left[t-\sqrt{\frac{\tau_0}{a}}(2nL+x)\right]\\ &+I_0\left[\frac{1}{2\tau_0}\sqrt{t^2-(2nL+2L-x)^2\frac{\tau_0}{a}}\right]u\left[t-\sqrt{\frac{\tau_0}{a}}(2nL+2L-x)\right]\end{aligned}\right\} \\ &\quad +\sqrt{\frac{a}{\tau_0}}\sum_{n=0}^{\infty}\int_0^t e^{-t'/2\tau_0}\left\{\begin{aligned}&I_0\left[\frac{1}{2\tau_0}\sqrt{t'^2-(2nL+x)^2\frac{\tau_0}{a}}\right]u\left[t'-\sqrt{\frac{\tau_0}{a}}(2nL+x)\right]\\ &+I_0\left[\frac{1}{2\tau_0}\sqrt{t'^2-(2nL+2L-x)^2\frac{\tau_0}{a}}\right]u\left[t'-\sqrt{\frac{\tau_0}{a}}(2nL+2L-x)\right]\end{aligned}\right\} \\ &\quad [\cos\omega(t-t')-\omega\tau_0\sin\omega(t-t')]\,dt'. \end{aligned} \tag{16}$$

For comparison with the solution of ref. [4], the same dimensionless quantities are introduced

$$V(X,Fo) = \frac{T(X,Fo)}{q_0L/\lambda}\quad Fo = \frac{at}{L^2}$$

$$Fo_1 = \frac{a}{\omega L^2} \quad Ve^2 = \frac{a\tau_0}{L^2} \quad X = \frac{x}{L}. \tag{17}$$

Then equation (16) is expressed as the following dimensionless form:

$$V(X,Fo) = Ve\,e^{-Fo/2Ve^2}\sum_{n=0}^{\infty}\left\{\begin{array}{l} I_0\left[\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+X)^2Ve^2}\right]u[Fo-(2n+X)Ve] \\ +I_0\left[\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+2-X)^2Ve^2}\right]u[Fo-(2n+2-X)Ve]\end{array}\right\}$$
$$+\frac{1}{Ve}\sum_{n=0}^{\infty}\int_0^{Fo} e^{-Fo'/2Ve^2}\left\{\begin{array}{l} I_0\left[\frac{1}{2Ve^2}\sqrt{Fo'^2-(2n+X)^2Ve^2}\right]u[Fo'-(2n+X)Ve] \\ +I_0\left[\frac{1}{2Ve^2}\sqrt{Fo'^2-(2n+2-X)^2Ve^2}\right]u[Fo'-(2n+2-X)Ve]\end{array}\right\}$$
$$\left[\cos\frac{Fo-Fo'}{Fo_1}-\frac{Ve^2}{Fo_1}\sin\frac{Fo-Fo'}{Fo_1}\right]dFo' \tag{18}$$

which takes a quite different form from that in ref. [4]. Similar to ref. [4], we consider a limit situation of the above solution, i.e. $\tau_0 \to 0$ (or $Ve \to 0$), and then the non-Fourier solution should go back to the Fourier solution. Under this condition, we have

$$\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+X)^2Ve^2} = \frac{Fo}{2Ve^2}-\frac{(2n+X)^2}{4Fo}+O(Ve^2) \tag{19a}$$

$$\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+2-X)^2Ve^2} = \frac{Fo}{2Ve^2}-\frac{(2n+2-X)^2}{4Fo}+O(Ve^2) \tag{19b}$$

$$\frac{1}{Ve\sqrt{2\pi\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+X)^2Ve^2}}} = \frac{1}{Ve\sqrt{2\pi\frac{1}{2Ve^2}\sqrt{Fo^2-(2n+2-X)^2Ve^2}}} = \frac{1}{\sqrt{\pi Fo}} \tag{19c}$$

and the Bessel function can be expanded into a series as follows:

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1+\frac{1}{8z}+\frac{9}{2!(8z)^2}+\cdots\right]. \tag{20}$$

By substituting equation (19) into equation (18) and employing equation (20), with $Ve \to 0$, the temperature distribution reduces to

$$V(X,Fo) = \sum_{n=0}^{\infty}\int_0^{Fo}\frac{1}{\sqrt{\pi F'o}}[e^{-(2n+X)^2/4F'o} + e^{-(2n+2-X)^2/4F'o}]\cos\frac{Fo-F'o}{Fo_1}dF'o. \tag{21}$$

This equation coincides with the Fourier solution which can be easily derived by using a Laplace transformation technique.

DISCUSSION

Numerical computations performed by using equation (18) to display the non-Fourier temperature profile in the medium show no difference with that from equation (17) of ref. [4], i.e. the two solutions predict the same temperature behaviors. Obviously it is not necessary for us to plot the temperature profiles once more by using the present solution.

From the analysis in the preceding section and ref. [4], it can be seen that when $\tau_0 \to 0$, the temperature solution obtained in this note reduces to equation (21), which can also be expressed as

$$V(X,Fo) = \int_0^{Fo}\left[\frac{1}{\sqrt{\pi Fo'}}\sum_{n=-\infty}^{\infty}e^{-(X+2n)^2/4Fo'}\right] \times \cos\frac{Fo-Fo'}{Fo_1}dFo' \tag{22}$$

and the other temperature solution reduces to equation (20) of ref. [4], which is equivalent to the following equation:

$$V(X,Fo) = \int_0^{Fo}\left\{1+2\sum_{n=1}^{\infty}\cos(n\pi X)e^{-n^2\pi^2 Fo'}\right\} \times \cos\frac{Fo-Fo'}{Fo_1}dFo'. \tag{23}$$

According to the generalized function theory [5], it is not difficult to prove the following relation:

$$1+2\sum_{n=1}^{\infty}\cos(n\pi X)e^{-n^2\pi^2 Fo} = \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\pi Fo}}e^{-(X+2n)^2/4Fo}. \tag{24}$$

Then we find that equations (22) and (23), i.e. the Fourier limits of the two solutions, are actually identical. Surely this does not imply that we can state with certainty that the two forms of the solutions, the analytical (ref. [4]) and the non-analytical (present note) one, are equivalent, although we are not able yet to prove this mathematically. From an engineering point of view, however, the numerical calculation and above discussion show that both of the solutions are available for predicting the temperature profiles.

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